0.1 General form

n equations, polynomials $p_1, \dots, p_n$, unknowns $x_1, \dots, x_n$, given variables $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n, d_1, \dots, d_n$,

Solve

$$\begin{array}{rcccl} p_1 & := & a_1x_1x_2 + b_1x_1 + c_1x_2 + d_1 & = & 0 \\ & & \vdots & & \\ p_{n-1} & := & a_{n-1}x_{n-1}x_n + b_{n-1}x_{n-1} + c_{n-1}x_n + d_n & = & 0 \\ p_n & := & a_nx_nx_1 + b_nx_n + c_nx_1 + d_n & = & 0 \end{array}$$

Solution:

Let $a_i^{(l+1)} = \begin{cases} a_i & l = 0 \\ a_{2i-1}^{(l)} & \lceil \frac{n}{2^{l-1}} \rceil \text{ is odd} \wedge i = \lceil \frac{n}{2^l} \rceil \\ \det \begin{pmatrix} a_{2i-1}^{(l)} & a_{2i}^{(l)} \\ b_{2i-1}^{(l)} & c_{2i}^{(l)} \end{pmatrix} & \text{else} \end{cases}$, $b_i^{(l+1)} = \begin{cases} b_i & l = 0 \\ b_{2i-1}^{(l)} & \lceil \frac{n}{2^{l-1}} \rceil \text{ is odd} \wedge i = \lceil \frac{n}{2^l} \rceil \\ \det \begin{pmatrix} a_{2i-1}^{(l)} & b_{2i}^{(l)} \\ b_{2i-1}^{(l)} & d_{2i}^{(l)} \end{pmatrix} & \text{else} \end{cases}$,

$c_i^{(l+1)} = \begin{cases} c_i & l = 0 \\ c_{2i-1}^{(l)} & \lceil \frac{n}{2^{l-1}} \rceil \text{ is odd} \wedge i = \lceil \frac{n}{2^l} \rceil \\ \det \begin{pmatrix} c_{2i-1}^{(l)} & a_{2i}^{(l)} \\ d_{2i-1}^{(l)} & c_{2i}^{(l)} \end{pmatrix} & \text{else} \end{cases}$, $d_i^{(l+1)} = \begin{cases} d_i & l = 0 \\ d_{2i-1}^{(l)} & \lceil \frac{n}{2^{l-1}} \rceil \text{ is odd} \wedge i = \lceil \frac{n}{2^l} \rceil \\ \det \begin{pmatrix} c_{2i-1}^{(l)} & b_{2i}^{(l)} \\ d_{2i-1}^{(l)} & d_{2i}^{(l)} \end{pmatrix} & \text{else} \end{cases}$.

Let $l = \lceil \log_2 n \rceil + 1$.

Then x_1 satisfies the equation

$$a_1^{(l)} x_1^2 + (b_1^{(l)} + c_1^{(l)}) x_1 + d_1^{(l)} = 0$$

Proof. **n = 2**:

$$\begin{aligned} & 0 \\ &= (a_2x_1 + b_2)p_1 - (a_1x_1 + c_1)p_2 \\ &= (a_2x_1 + b_2)(a_1x_1x_2 + b_1x_1 + c_1x_2 + d_1) - (a_1x_1 + c_1)(a_2x_2x_1 + b_2x_2 + c_2x_1 + d_2) \\ &= (a_2x_1 + b_2)(x_2(a_1x_1 + c_1) + b_1x_1 + d_1) - (a_1x_1 + c_1)(x_2(a_2x_1 + b_2) + c_2x_1 + d_2) \\ &= (a_2x_1 + b_2)(b_1x_1 + d_1) - (a_1x_1 + c_1)(c_2x_1 + d_2) \\ &= x_1^2(a_2b_1 - a_1c_2) + x_1(a_2d_1 + b_2b_1 - a_1d_2 - c_1c_2) + b_2d_1 - c_1d_2 \end{aligned}$$

n = 3:

$$
\begin{aligned}
&0\\
&= -(x_3a_2+b_2)(x_1a_3+b_3)p_1 + (x_1a_1+c_1)(x_1a_3+b_3)p_2 + (a_2(x_1b_1+d_1) - c_2(x_1a_1+c_1))p_3\\
&= -(x_3a_2+b_2)(x_1a_3+b_3)(a_1x_1x_2+b_1x_1+c_1x_2+d_1)\\
&\quad + (x_1a_1+c_1)(x_1a_3+b_3)(a_2x_2x_3+b_2x_2+c_2x_3+d_2)\\
&\quad + (a_2(x_1b_1+d_1) - c_2(x_1a_1+c_1))(a_3x_3x_1+b_3x_3+c_3x_1+d_3)\\
&= -(x_3a_2+b_2)(x_1a_3+b_3)((a_1x_1+c_1)x_2+b_1x_1+d_1)\\
&\quad + (x_1a_1+c_1)(x_1a_3+b_3)(x_2(a_2x_3+b_2)+c_2x_3+d_2)\\
&\quad + (a_2(x_1b_1+d_1) - c_2(x_1a_1+c_1))(a_3x_3x_1+b_3x_3+c_3x_1+d_3)\\
&= -(x_3a_2+b_2)(x_1a_3+b_3)(b_1x_1+d_1)\\
&\quad + (x_1a_1+c_1)(x_1a_3+b_3)(c_2x_3+d_2)\\
&\quad + (a_2(x_1b_1+d_1) - c_2(x_1a_1+c_1))(a_3x_3x_1+b_3x_3+c_3x_1+d_3)\\
&= -(x_3a_2+b_2)(x_1a_3+b_3)(b_1x_1+d_1)\\
&\quad + (x_1a_1+c_1)(x_1a_3+b_3)(c_2x_3+d_2)\\
&\quad + (a_2(x_1b_1+d_1) - c_2(x_1a_1+c_1))(x_3(a_3x_1+b_3)+c_3x_1+d_3)\\
&= (x_1a_3+b_3)\left(-(x_3a_2+b_2)(b_1x_1+d_1) + (x_1a_1+c_1)(c_2x_3+d_2) + (a_2(x_1b_1+d_1) - c_2(x_1a_1+c_1))x_3\right)\\
&\quad + (a_2(x_1b_1+d_1) - c_2(x_1a_1+c_1))(c_3x_1+d_3)\\
&= (x_1a_3+b_3)\left(-(x_3a_2+b_2)(b_1x_1+d_1) + (x_1a_1+c_1)(d_2) + (a_2(x_1b_1+d_1))x_3\right)\\
&\quad + (a_2(x_1b_1+d_1) - c_2(x_1a_1+c_1))(c_3x_1+d_3)\\
&= (x_1a_3+b_3)\left(-(b_2)(b_1x_1+d_1) + (x_1a_1+c_1)(d_2)\right)\\
&\quad + (a_2(x_1b_1+d_1) - c_2(x_1a_1+c_1))(c_3x_1+d_3)\\
&= \qquad x_1^2(-a_3b_1b_2 + a_1a_3d_2 + a_2b_1c_3 - a_1c_2c_3)\\
&\quad + x_1(-a_3b_2d_1 + a_3c_1d_2 - b_1b_2b_3 + a_1b_3d_2 + a_2b_1d_3 - a_1c_2d_3 + a_2b_1c_3 - c_1c_2c_3)\\
&\quad - b_2b_3d_1 + b_3c_1d_3 + a_2d_1d_3 - c_1c_2d_3
\end{aligned}
$$

$\mathbf{n > 3}$

Let $x_0 = x_n, x_{n+1} = x_1, a_0 = a_n, a_{n+1} = a_1, b_0 = b_n, b_{n+1} = b_1, c_0 = c_n, c_{n+1} = c_1, d_0 = d_n, d_{n+1} = d_1$.

Let $p'_i = \begin{cases} (x_{2i+1}a_{2i}+b_{2i})p_{2i-1} - (x_{2i-1}a_{2i-1}+c_{2i-1})p_{2i} & 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ p_n & i = \lfloor \frac{n}{2} \rfloor + 1, n \text{ odd} \\ \bot & \text{else} \end{cases}$.

$$
\begin{aligned}
&p'_i\\
&= (x_{2i+1}a_{2i}+b_{2i})p_{2i-1} - (x_{2i-1}a_{2i-1}+c_{2i-1})p_{2i}\\
&= (x_{2i+1}a_{2i}+b_{2i})(a_{2i-1}x_{2i-1}x_{2i}+b_{2i-1}x_{2i-1}+c_{2i-1}x_{2i}+d_{2i-1}) - (x_{2i-1}a_{2i-1}+c_{2i-1})(a_{2i}x_{2i}x_{2i+1}+b_{2i}x_{2i}+c_{2i}x_{2i+1}+d\\
&= (x_{2i+1}a_{2i}+b_{2i})(x_{2i}(a_{2i-1}x_{2i-1}+c_{2i-1})+b_{2i-1}x_{2i-1}+d_{2i-1}) - (x_{2i-1}a_{2i-1}+c_{2i-1})(x_{2i}(a_{2i}x_{2i+1}+b_{2i})+c_{2i}x_{2i+1}+d_{2i})\\
&= (x_{2i+1}a_{2i}+b_{2i})(b_{2i-1}x_{2i-1}+d_{2i-1}) - (x_{2i-1}a_{2i-1}+c_{2i-1})(c_{2i}x_{2i+1}+d_{2i})\\
&= x_{2i-1}x_{2i+1}(a_{2i}b_{2i-1} - a_{2i-1}c_{2i}) + x_{2i-1}(b_{2i-1}b_{2i} - a_{2i-1}d_{2i}) + x_{2i+1}(a_{2i}d_{2i-1} - c_{2i-1}c_{2i}) + d_{2i-1}b_{2i} - c_{2i-1}d_{2i}
\end{aligned}
$$

with $x'_i = x_{2i-1}, x'_{i+1} = x_{2i+1}, a'_i = \det\begin{pmatrix} a_{2i-1} & a_{2i} \\ b_{2i-1} & c_{2i}\end{pmatrix}, b'_i = \det\begin{pmatrix} a_{2i-1} & b_{2i} \\ b_{2i-1} & d_{2i}\end{pmatrix}, c'_i = \det\begin{pmatrix} c_{2i-1} & a_{2i} \\ d_{2i-1} & c_{2i}\end{pmatrix}, d'_i =$

$\det \begin{pmatrix} c_{2i-1} & b_{2i} \\ d_{2i-1} & d_{2i} \end{pmatrix}$

For even $n = 2l$, new equations $p'_1, \ldots, p'_l$, for odd $n = 2l + 1$ new equations $p'_1, \ldots, p'_l, p_{2l+1}$

$\square$