# Constructing Separators and Adjustment Sets in Ancestral Graphs

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# Motivation

- Given any causal graph $G$, any set of exposures $\mathbf{X}$ and any set of outcomes $\mathbf{Y}$ we want to calculate the causal effect of $\mathbf{X}$ on $\mathbf{Y}$ with a statistically well understood and robust method.
- Standard technique is to use a set $\mathbf{Z}$ of covariate adjustments such that the causal effect is given by

$$\sum_{z} p(y \mid x, z) p(z)$$

- How can we test, find or enumerate all (minimal/minimum) adjustment sets?

# Overview

# Previous Work

- Backdoor criterion [Pea09]
- Frontdoor criterion [Pea09]
- Do-calculus [Pea09]
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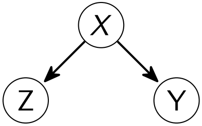

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  Not directly applicable in algorithms

# Method

- Reduce adjustment to d-separation
- Use d-separation algorithms

# Adjustment in DAGs: criterion

**Theorem (Adjustment criterion, simplifying [SVR10, Shp12])**

*A set* $\mathbf{Z}$ *is an adjustment relative to sets* $\mathbf{X}$ *and* $\mathbf{Y}$ *if and only if:*

1. $\mathbf{Z}$ *does not block any causal path and* $\mathbf{X}(\rightarrow, \rightarrow\rightarrow, \ldots)\mathbf{Y}$
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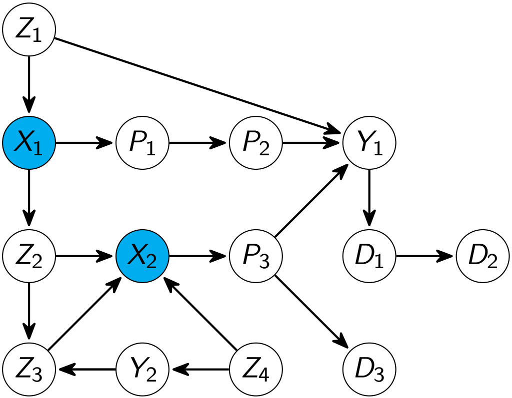

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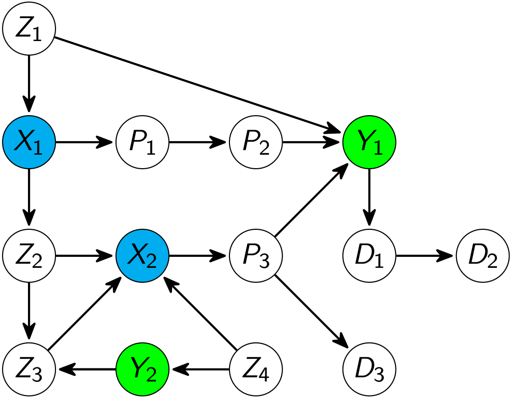

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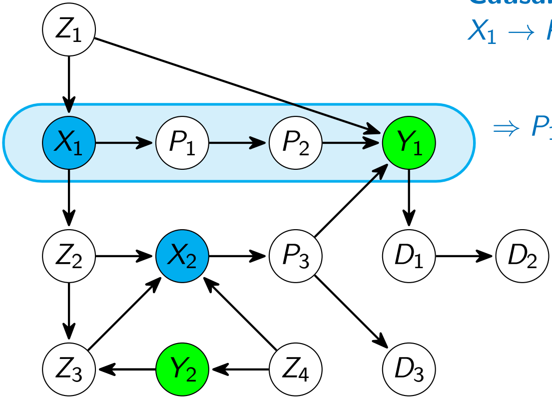


**Causal paths:**
$X_1 \rightarrow P_1 \rightarrow P_2 \rightarrow Y_1$

$\Rightarrow P_1, P_2, P_3 \notin \mathbf{Z}$

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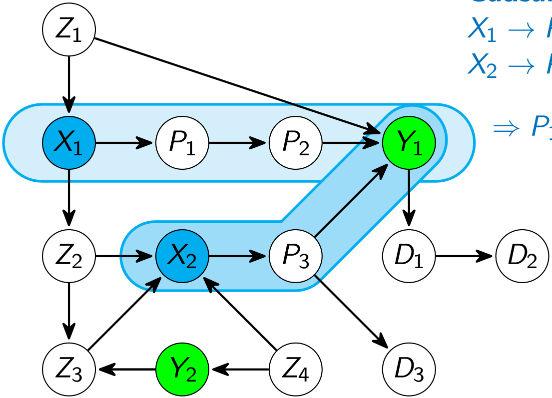


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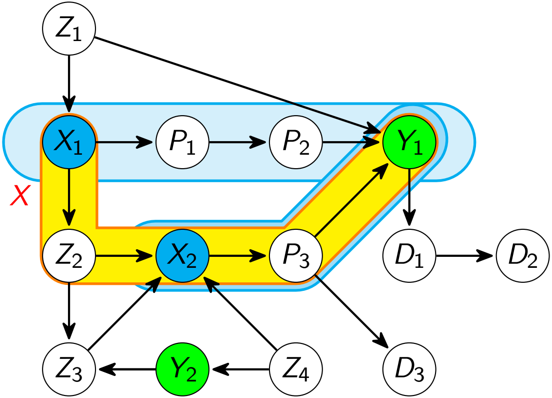
Causal
but too many $X$
$\Rightarrow$ irrelevant



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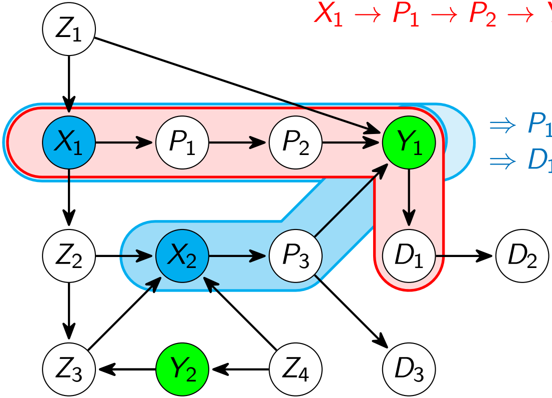
**Possible noncausal paths:**

$X_1 \rightarrow P_1 \rightarrow P_2 \rightarrow Y_1 \rightarrow D_1 \leftarrow Y_1$

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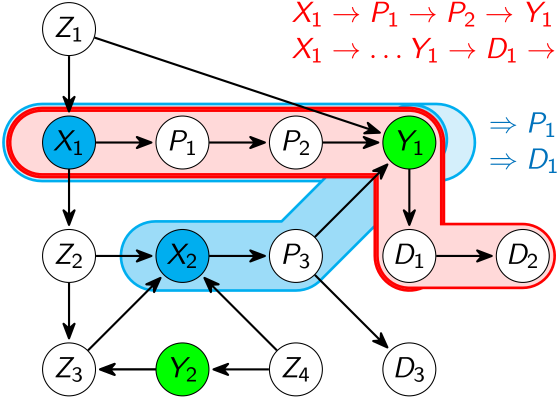


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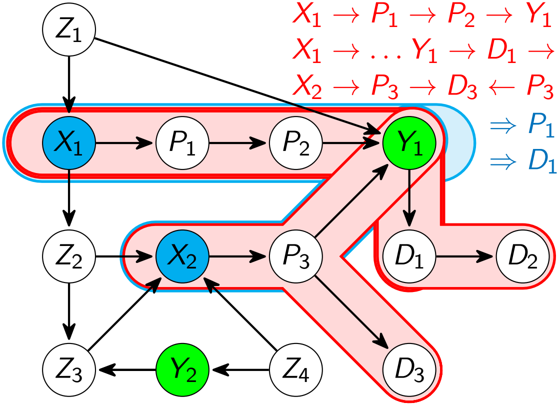

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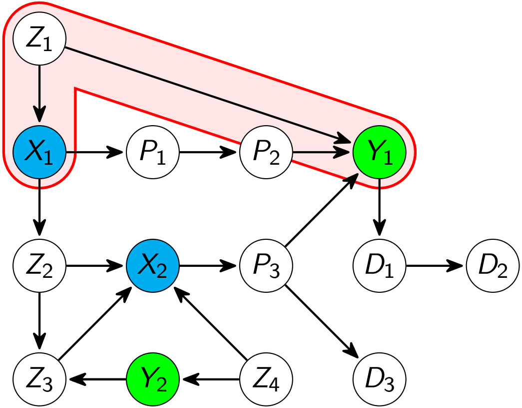

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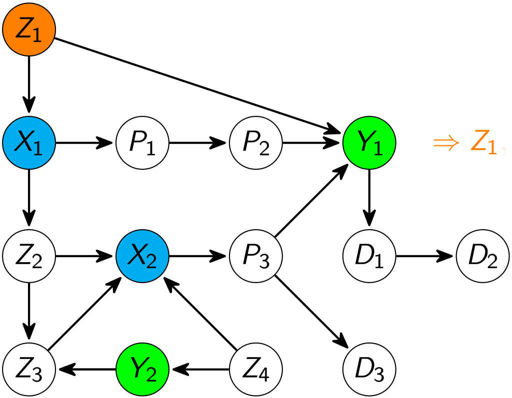


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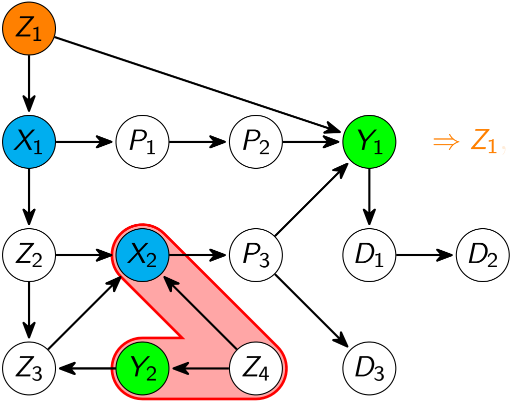


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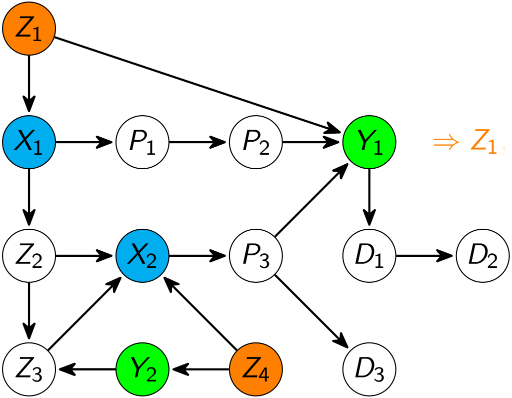


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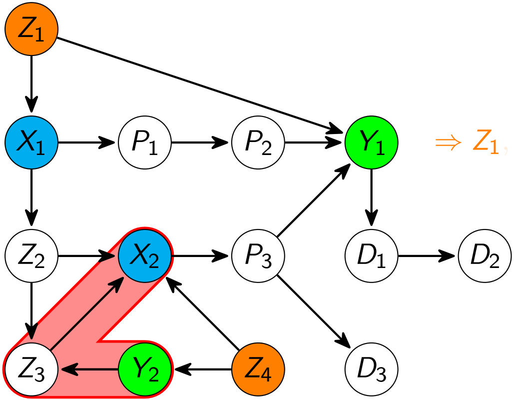


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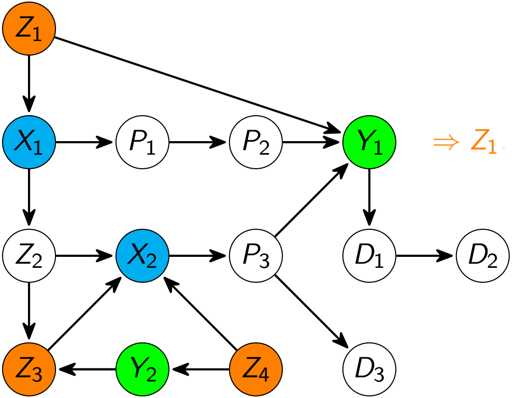


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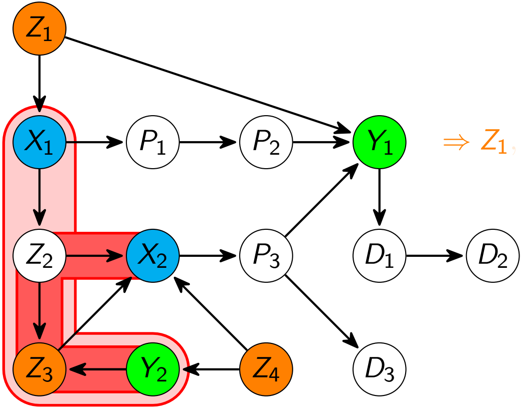


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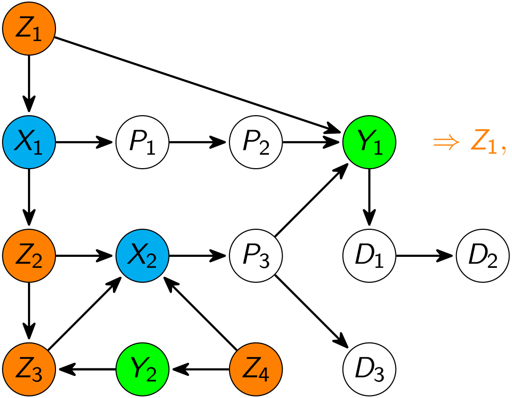


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**Theorem (New adjustment criterion)**

*A set* $\mathbf{Z}$ *is an adjustment relative to sets* $\mathbf{X}$ *and* $\mathbf{Y}$ *if and only if:*

1. $\mathbf{Z}$ *neither contain nodes of causal paths nor their descendants and*
2. $\mathbf{Z}$ *separates* $\mathbf{X}$ *and* $\mathbf{Y}$ *in a modified graph without causal paths*

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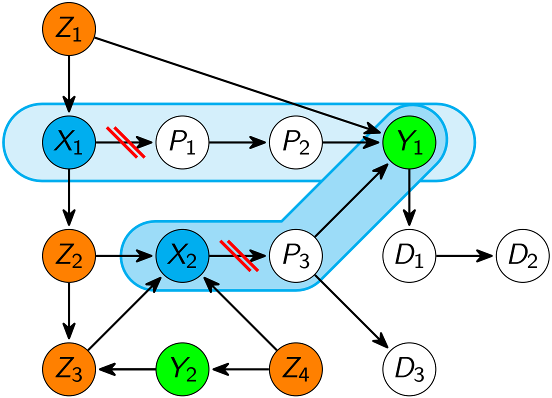

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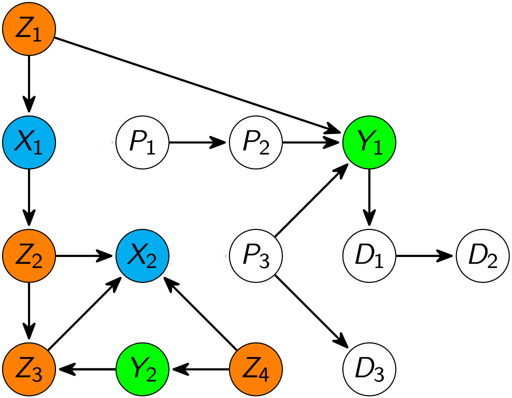


# Algorithms for d-separation

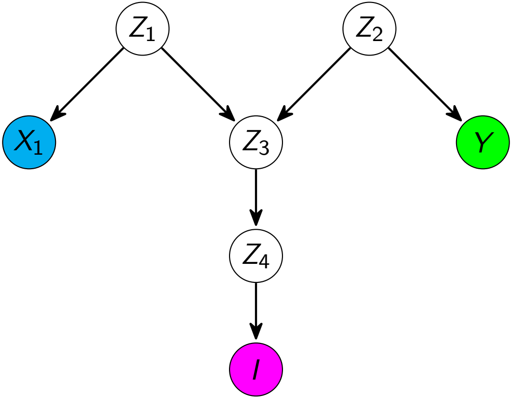
Need to generalize existing separation algorithms to

- support sets $|\mathbf{X}| > 1$
- find only $\mathbf{Z}$ restricted to $\mathbf{Z} \subseteq \mathbf{R}$
- (find only $\mathbf{Z}$ containing $\mathbf{I} \subseteq \mathbf{Z}$ to enumerate them all)

# Algorithms for d-separation: Ideas

## Finding any separators

Ancestors of $\mathbf{X}, \mathbf{Y}, \mathbf{I}$ are separator or none exists




$O(n + m)$

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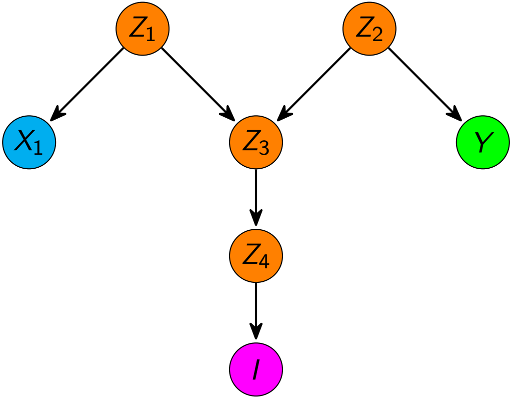


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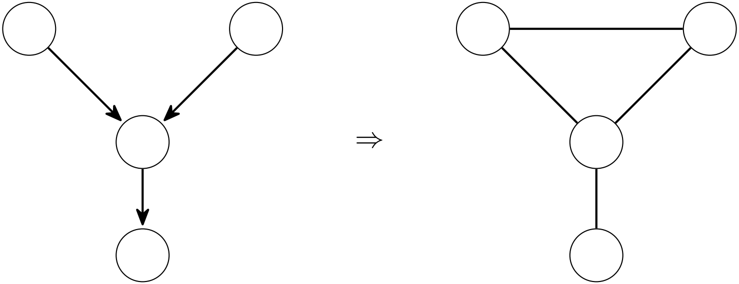
# Algorithms for d-separation: Ideas

## Finding minimal separators, generalizing [AdC96, TPP98, RS02]

Moralization:
"Marry" parents to create undirected graph:



$O(n^2)$

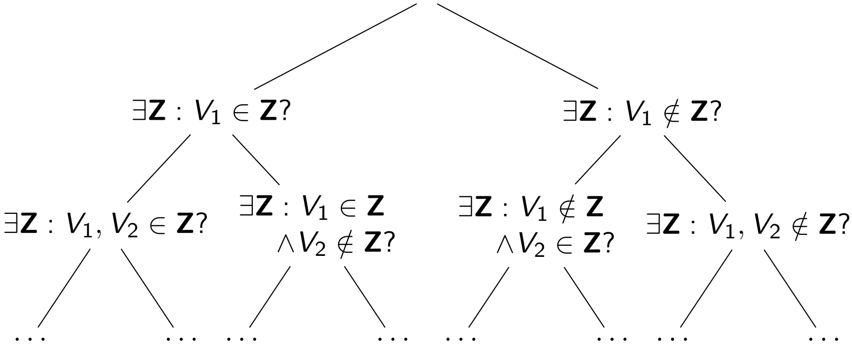
Find minimal vertex cut with standard search

Find minimum vertex cut with min-cut-max-flow. $O(n^3)$

# Algorithms for d-separation: Ideas

## Enumerating all separators, idea from [Tak10, TL11]

Prune search tree:

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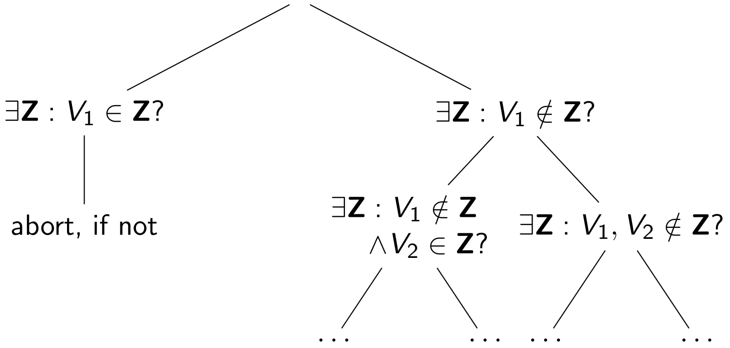


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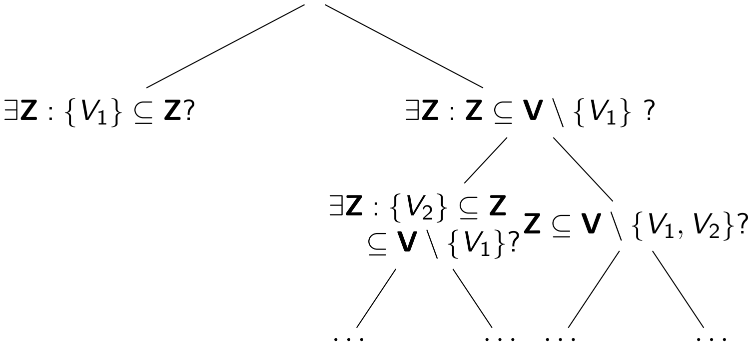


$O(n^3)$ delay

# Algorithms for adjustments

**Verification:** For given $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ decide if ...

TestAdj $\mathbf{Z}$ is an adjustment to $\mathbf{X}, \mathbf{Y}$ — $\mathcal{O}(n+m)$

TestMinAdj $\mathbf{Z}$ is an adjustment to $\mathbf{X}, \mathbf{Y}$ but no $\mathbf{Z}' \subsetneq \mathbf{Z}$ does — $\mathcal{O}(n^2)$

**Construction:** For given $\mathbf{X}, \mathbf{Y}$ and auxiliary $\mathbf{I}, \mathbf{R}$, output ...

FindAdj an adjustment $\mathbf{Z}$ with $\mathbf{I} \subseteq \mathbf{Z} \subseteq \mathbf{R}$ — $\mathcal{O}(n+m)$

FindMinAdj a minimal adjustment $\mathbf{Z}$ with $\mathbf{I} \subseteq \mathbf{Z} \subseteq \mathbf{R}$ — $\mathcal{O}(n^2)$

FindMinCostAdj a minimum-cost adjustment $\mathbf{Z}$ with $\mathbf{I} \subseteq \mathbf{Z} \subseteq \mathbf{R}$ — $\mathcal{O}(n^3)$

**Enumeration:** For given $\mathbf{X}, \mathbf{Y}, \mathbf{I}, \mathbf{R}$ enumerate all ...

ListAdj adjustments $\mathbf{Z}$ with $\mathbf{I} \subseteq \mathbf{Z} \subseteq \mathbf{R}$ — $\mathcal{O}(n(n+m))$ delay

ListMinAdj minimal adjustments $\mathbf{Z}$ with $\mathbf{I} \subseteq \mathbf{Z} \subseteq \mathbf{R}$ — $\mathcal{O}(n^3)$ delay

# Generalization

## Ancestral graphs

Generalized DAGs with additional $-$ and $\leftrightarrow$ edges:

$$X - Y \text{ similar to } X \to S \leftarrow Y$$
$$X \leftrightarrow Y \text{ similar to } X \leftarrow L \to Y$$

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Generalized DAGs with additional $-$ and $\leftrightarrow$ edges:

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$\Rightarrow$ Separation algorithms can be used

# Generalization

## Maximal ancestral graphs (MAGs)

- MAGs are used, when not all causal variables are known.
- All non separable nodes must be adjacent.
- Multiple DAGs $G$ can be represented by a single MAG $G[^{\mathbf{L}}_{\mathbf{S}}$.

**Example**

MAG: $G[^{\mathbf{L}}_{\mathbf{S}} = X \to Y$

Represents among others DAGs:

$G_1 = X \to Y$ with $\mathbf{L} = \mathbf{S} = \emptyset$

$G_2 = X \leftarrow L \rightarrow Y,\ X \to Y$ with $\mathbf{L} = L$, $\mathbf{S} = \emptyset$

- $\mathbf{Z}$ is an adjustment in MAG $G[^{\mathbf{L}}_{\mathbf{S}}$, if (and only if) it is an adjustment in every represented DAG $G$

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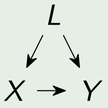
$G_2 =$  with $\mathbf{L} = L$, $\mathbf{S} = \emptyset$

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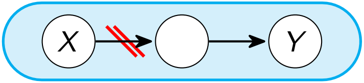
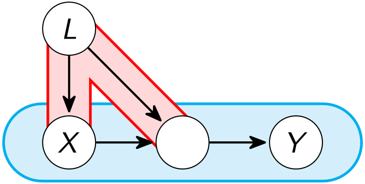
- If any of the removed edges of the

MAG $G[^{\mathbf{L}}_{\emptyset}$:




corresponds to a non-causal path in any

DAG $G$ :




no adjustment set exists.

- Otherwise, the adjustment criterion works as usual
  $\Rightarrow$ all algorithms for adjustment can be used on MAGs
- Correspondence testable in $O(|\text{children of } \mathbf{X}|(n+m))$

# Conclusion

- We have shown an equivalence between separators and adjustment sets
- We provide a collection of algorithms that
  - solve the problems of
    - verifying
    - finding
    - enumerating

    arbitrary, minimal and minimum
    - d-separators in DAGs
    - m-separators in AGs
    - adjustment sets in DAGs and MAGs
  - are easy to implement

Questions?

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